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THE TRANSMISSION OF SOUND THROUGH AN INHOMOGENEOUS ANISOTROPIC LAYER ADJOINING VISCOUS LIQUIDS†

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The problem of the reflection and transmission of a plane sound wave through a plane inhomogeneous elastic layer, the material of which possesses a general form of anisotropy while the liquids adjoining it are viscous, is considered. An analytic description of the acoustic fields outside the layer is obtained. The boundary-value problem for the system of ordinary differential equations, constructed in order to determine the displacement field in the elastic layer, is reduced to problems with initial conditions. The absorption coefficient is calculated numerically. © 1999 Elsevier Science Ltd. All rights reserved.

The reflection and transmission of sound waves through a plane inhomogeneous isotropic elastic layer [1], a homogeneous transversely isotropic elastic layer [2, 3], an inhomogeneous transversely isotropic elastic layer [4] were analysed previously. In all these papers it was assumed that the elastic layer has ideal liquids adjoining it.

1. Consider an inhomogeneous anisotropic plane layer of thickness $2h$, the moduli of elasticity and the density of the material of which are described by continuous differential functions of the coordinate x_3. A system of rectangular coordinates x_1, x_2, x_3 is chosen in such a way that the x_1 axis lies in the middle plane of the layer, while the x_3 axis is directed downwards along the normal to the layer surface. We will assume that the upper and lower surfaces of the layer are bounded by viscous homogeneous liquids, which have densities ρ_1 and ρ_2, velocities of sound c_1 and c_2 and kinematic coefficients of viscosity ν_1 and ν_2, respectively.

Suppose a plane sound wave, the velocity potential of which is

$$\Psi_i = \exp\{i[k_{11}x_1 + k_{13}(x_3 + h) - \omega t]\} \tag{1.1}$$

is incident on the elastic layer from the half-space $x_3 < -h$, where $k_{11} = k_1 \sin\theta_1$, $k_{13} = k_1 \cos\theta$ are the projections of the wave vector $\mathbf{k}_1$ onto the coordinate axes x_1 and x_3 respectively, $k_1 = \omega/c_1$ is the wave number in the upper half-space, ω is the angular frequency and θ_1 is the angle of incidence of the plane wave. The time factor $\exp(-i\omega t)$ is henceforth omitted.

We will determine the wave reflected from the layer and passing through it and we will also obtain the displacement field inside the elastic layer.

2. The propagation of elastic waves in an inhomogeneous anisotropic layer is described by the general equations of motion of an elastic medium [5]

$$\rho\ddot{u}_i = \partial\sigma_{ij}/\partial x_j, \quad i = 1, 2, 3 \tag{2.1}$$

where $\rho = \rho(x_3)$ is the density of the layer material, u_i is the projection of the displacement vector u onto the x_i axis, and σ_{ij} are the components of the stress tensor, which, in the general case of anisotropy, are related to the components of the strain tensor ε_{kl} as follows:

$$\sigma_{ij} = \lambda_{ijkl}\varepsilon^{kl}, \quad \varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) \tag{2.2}$$

Here $\lambda_{ijkl} = \lambda_{ijkl}(x_3)$ are the moduli of elasticity of the anisotropic material.

Henceforth we will use a double subscript for the moduli of elasticity λ_{ik}, where $i, k = 1, 2, \ldots, 6$.

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Here the values of the subscripts 1, 2, ..., 6 denote the pair of indices 11, 22, 33, 13, and 12, respectively.

Since the wave vector of the incident wave lies in the x_1, x_3 plane and, consequently, the exciting field is independent of the x_2 coordinate, while the inhomogeneity of the material of the layer manifests itself only along the x_3 axis, neither the reflected field, the field transmitted into the half-space $x_3 > h$, nor the field excited in the elastic layer should depend on the x_2 coordinate. Note also that, by Snell's law [6], the dependence of the components of the displacement vector on the x_1 coordinate will have the form $\exp(ik_{11}x_1)$. Hence, we will seek the projections of the displacement vector in the form

$$u_j = U_j(x_3)\exp(ik_{11}x_1) \tag{2.3}$$

In addition to the unknown quantities u_1, u_2 and u_3, we will introduce the components of the stress tensor σ_{13}, σ_{23}, σ_{33} as additional unknowns, characterizing the motion of the elastic medium. On the basis of the considerations given above, we will represent these components in the form

$$\sigma_{j3}(x_1, x_3) = \sigma_j(x_3)\exp(ik_{11}x_1), \quad j = 1, 2, 3 \tag{2.4}$$

We will obtain equations for determining the unknown quantities $U_j(x_3)$ and $\sigma_j(x_3)$ ($j = 1, 2, 3$). We substitute expressions (2.3) into Hooke's relations (2.2). From the equations obtained we then express the derivatives U_j' with respect to the coordinate x_3 in terms of the function σ_1, σ_2, σ_3. Combining the expressions obtained with the equation of motion (2.1), we obtain, after substituting (2.3) and (2.4) into it, the following system of first-order linear ordinary differential equations in the unknowns U_1, U_2, U_3 and σ_1, σ_2, σ_3

$$\begin{aligned} \mathbf{U}' &= -sA^{-1}B\mathbf{U} + A^{-1}\mathbf{P} \\ \mathbf{P}' &= [-s^2(C - B^TA^{-1}B) - \omega^2\rho I]\mathbf{U} - sB^T\mathbf{P} \end{aligned} \tag{2.5}$$

Here

$$\mathbf{U} = (U_1, U_2, U_3)^T, \quad \mathbf{P} = (\sigma_1, \sigma_2, \sigma_3)^T, \quad s = ik_{11}$$

$$A = \begin{Vmatrix} \lambda_{55} & \lambda_{54} & \lambda_{53} \\ \lambda_{45} & \lambda_{44} & \lambda_{43} \\ \lambda_{35} & \lambda_{34} & \lambda_{33} \end{Vmatrix}, \quad B = \begin{Vmatrix} \lambda_{51} & \lambda_{56} & \lambda_{55} \\ \lambda_{41} & \lambda_{46} & \lambda_{45} \\ \lambda_{31} & \lambda_{36} & \lambda_{35} \end{Vmatrix}, \quad C = \begin{Vmatrix} \lambda_{11} & \lambda_{16} & \lambda_{15} \\ \lambda_{61} & \lambda_{66} & \lambda_{65} \\ \lambda_{51} & \lambda_{56} & \lambda_{55} \end{Vmatrix}$$

and I is the identity matrix.

Note that the use of the components of the strain tensor σ_{j3} ($j = 1, 2, 3$) as the unknown functions has led to differential equations, the coefficients of which do not contain derivatives of the moduli of elasticity. This enables us, when solving this problem, to confine ourselves to the requirement that the moduli of elasticity and the density of the layer material should be continuous.

We will represent the velocity of liquid particles in the sound waves reflected from and transmitted through the layer in the form

$$\mathbf{v}_1 = \operatorname{grad}\Psi_1 + \operatorname{rot}\mathbf{\Phi}_1, \quad \mathbf{v}_2 = \operatorname{grad}\Psi_2 + \operatorname{rot}\mathbf{\Phi}_2$$

The velocity potentials of the longitudinal waves Ψ_1, Ψ_2 and the vector potentials of the velocities of the viscous waves $\mathbf{\Phi}_1$, $\mathbf{\Phi}_2$ (in the upper and lower half-spaces, respectively) are solutions of the following equations

$$\Delta\Psi_j + k_j^2\Psi_j = 0 \tag{2.6}$$

$$\Delta\mathbf{\Phi}_j + \varkappa_j^2\mathbf{\Phi}_j = 0, \quad j = 1, 2 \tag{2.7}$$

where $k_j = \omega/c_j$ is the wave number of the longitudinal waves and $\varkappa_j = i\omega/\nu_j$ is the wave number of the viscous waves.

Note that Eq. (2.7) consists of a system of three scalar equations in the three functions Φ_{j1}, Φ_{j2}, Φ_{j3}, where Φ_{ji} is the projection of the vector $\mathbf{\Phi}_j$ onto the x_i axis ($i = 1, 2, 3$). However, the equation $\operatorname{div}\mathbf{\Phi}_j = 0$ ($j = 1, 2$) gives the relation between the three functions Φ_{j1}, Φ_{j2}, Φ_{j3} and of these three functions two remain linearly independent.

We will consider the functions Ψ_1, Ψ_2 and Φ_{12}, Φ_{22} as the unknowns. In addition to these, we introduce the additional unknowns υ_{12} and υ_{22} (the projections of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ onto the x_2 axis).

We will obtain equations for the functions $\boldsymbol{\Phi}_{j2}$ and υ_{j2} ($j = 1, 2$). For steady motion, from the linearized Navier–Stokes and continuity equations we obtain the following equation in the velocity vector $\mathbf{v}_j$

$$-\operatorname{rot}\operatorname{rot}\mathbf{v}_j + \frac{\varkappa_j^2}{k_j^2}\operatorname{grad}\operatorname{div}\mathbf{v}_j + \varkappa_j^2\mathbf{v}_j = 0, \quad j = 1,2 \tag{2.8}$$

We project the vector equations (2.7) and (2.8) onto the x_2 coordinate axis, taking into account the fact that $\boldsymbol{\Phi}_j = \boldsymbol{\Phi}_j(x_1, x_3)$ and $\mathbf{v} = \mathbf{v}(x_1, x_3)$. As a result, we obtain two identical equations

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2} + \varkappa_j^2\right)\begin{Bmatrix}\Phi_{j2}\\ \upsilon_{j2}\end{Bmatrix} = 0, \quad j = 1,2$$

Taking Snell's law into account, we will seek the functions Ψ_j, Φ_{j2}, υ_{j2} ($j = 1, 2$) in the form

$$\begin{gathered}\Psi_j = A_{j1}\exp[ik_{11}x_1 \mp ik_{j3}(x_3 \pm h)], \quad \Phi_{j2} = A_{j2}\exp[ik_{11}x_1 \mp i\varkappa_{j3}(x_3 \pm h)]\\ \upsilon_{j2} = A_{j3}\exp[ik_{11}x_1 \mp i\varkappa_{j3}(x_3 \pm h)]\end{gathered} \tag{2.9}$$

where the upper signs are taken when $j = 1$ and the lower signs when $j = 2$, and $k_{j3} = \sqrt{(k_j^2 - k_{j1}^2)}$ and $\varkappa_{j3} = \sqrt{(\varkappa_j^2 - k_{j1}^2)}$ are the projections of the wave vectors $\mathbf{k}_j$ and $\varkappa_2$ onto the x_3 coordinate axis.

The coefficients A_{jk} ($j = 1, 2$; $k = 1, 2, 3$) remain to be determined from the boundary conditions, which consist of the continuity of the velocities and directions on both surfaces of the elastic layer. We have

$$\begin{gathered}x_3 = -h:\ -i\omega\mathbf{u} = \mathbf{v}, \quad \sigma_{13} = \sigma_{13}^{(1)}, \quad \sigma_{23} = \sigma_{23}^{(1)}, \quad \sigma_{33} = \sigma_{33}^{(1)}\\ x_3 = h:\ -i\omega\mathbf{u} = \mathbf{v}_2, \quad \sigma_{13} = \sigma_{13}^{(2)}, \quad \sigma_{23} = \sigma_{23}^{(2)}, \quad \sigma_{33} = \sigma_{33}^{(2)}\end{gathered} \tag{2.10}$$

where $\mathbf{v} = \operatorname{grad}\Psi_i + \mathbf{v}_1$ is the velocity of liquid particles in the upper half-space, $p_1 = i\omega\rho_1(\Psi_i + \Psi_1)$, $p_2 = i\omega\rho_2\Psi_2$ are the acoustic pressures in the upper and lower half-spaces, and $\sigma_{kl}^{(j)}$ are the components of the stress tensor of the viscous liquids in the upper half-space ($j = 1$) and the lower half-space ($j = 2$). In this case

$$\sigma_{kl}^{(j)} = -p_j\delta_{kl} + \rho_j\left[\nu_j\left(\frac{\partial \upsilon_{jl}}{\partial x_k} + \frac{\partial \upsilon_{jk}}{\partial x_l}\right) + \delta_{kl}\left(\nu_j' - \frac{2}{3}\nu_j\right)\operatorname{div}\mathbf{v}_j\right] \tag{2.11}$$

where ν_j' is the second kinematic coefficient of viscosity of the jth liquid.

We substitute expressions (1.1), (2.3), (2.4), (2.9) and (2.11) into boundary conditions (2.10). We thereby obtain

$$\begin{aligned}A_{11} &= [k_{13}\varkappa_{13} - k_{11}^2 + \omega(\varkappa_{13}U_3 - k_{11}U_1)]/\Delta_1\,|_{x_3=-h}\\ A_{12} &= -[2k_{11}k_{13} + \omega(k_{13}U_1 + k_{11}U_3)]/\Delta_1\,|_{x_3=-h}\\ A_{13} &= -i\omega U_2\,|_{x_3=-h}\\ A_{21} &= -\omega(k_{11}U_1 + \varkappa_{23}U_3)/\Delta_2\,|_{x_3=h}\\ A_{22} &= \omega(k_{23}U_1 - k_{11}U_3)/\Delta_2\,|_{x_3=h}, \quad A_{23} = -i\omega U_2\,|_{x_3=h}\end{aligned} \tag{2.12}$$

and six conditions for finding a particular solution of system (2.5)

$$[\mathbf{P} + E_1\mathbf{U}]_{x_3=-h} = \mathbf{D}, \quad [\mathbf{P} + E_2\mathbf{U}]_{x_3=h} = 0 \tag{2.13}$$

where

$$E_j = \frac{\omega\rho_j\nu_j}{\Delta_j}\begin{Vmatrix} \pm k_{j3}\varkappa_j^2 & 0 & k_{11}(\varkappa_j^2 - 2\Delta_j) \\ 0 & \pm\varkappa_{j3}\Delta_j & 0 \\ k_{11}(2\Delta_j - a_j) & 0 & \pm\varkappa_{j3}a_j \end{Vmatrix}$$

$$\mathbf{D} = -2k_{13}\frac{\rho_1\nu_1}{\Delta_1}(k_{11}\varkappa_1^2, 0, \varkappa_{13}a_1)^T$$

$$\Delta_j = k_{11}^2 + k_{j3}\varkappa_{j3}, \quad a_j = \varkappa_j^2 + k_j^2\left(\frac{4}{3} + \frac{\nu_j'}{\nu_j}\right), \quad j = 1,2$$

Note that the coefficients A_{jk} can be calculated after determining the values of the functions U_1, U_2 and U_3 on the surfaces of the layer (with $x_3 = -h$ and $x_3 = h$).

3. To find the functions U_1, U_2 and U_3 we need to solve boundary-value problem (2.5), (2.13).

We reduce this boundary-value problem to a problem with initial conditions. We obtain three linearly independent solutions of the system of differential equations (2.5) $\mathbf{U}_k$, $\mathbf{P}_k$ ($k = 1, 2, 3$) which satisfy the second boundary condition (2.13). We choose the following as the initial conditions for them

$$\mathbf{U}_k|_{x_3=h} = (\delta_{1k}, \delta_{2k}, \delta_{3k})^T, \quad \mathbf{P}_k|_{x_3=h} = -E_2\mathbf{U}_k|_{x_3=h}, \quad k = 1,2,3 \tag{3.1}$$

The Cauchy problem (2.5), (3.1) can be solved by one of the numerical methods.

Obviously a solution of system (2.5), which satisfies the second boundary condition of (2.13), will be any linear combination made up of the solutions of these Cauchy problems

$$\mathbf{U} = \sum_{k=1}^{3} C_k\mathbf{U}_k, \quad \mathbf{P} = \sum_{k=1}^{3} C_k\mathbf{P}_k \tag{3.2}$$

where C_1, C_2 and C_3 are arbitrary constants. By choosing these coefficients we can satisfy the first boundary condition of (2.13). Substituting (3.2) into it, we obtain a system of linear algebraic equations in the unknowns C_1, C_2 and C_3

$$\sum_{k=1}^{3} C_k(\mathbf{P}_k + E_1\mathbf{U}_k)|_{x_3=-h} = \mathbf{D}$$

Solving this system, we obtain the coefficients C_1, C_2 and C_3. We determine the functions U_1, U_2 and U_3 using (3.2), and from (2.12) we obtain the reflection coefficient A_{1k} and the transmission coefficient A_{2k}. As a result, we obtain an analytic description of the acoustic fields outside the layer. The displacement and stress fields inside the elastic layer are found using the functions $\mathbf{U}(x_3)$ and $\mathbf{P}(x_3)$, which, in the case of a numerical solution of boundary-value problem (2.5), (2.13), will be discrete sets of values at points of the integration interval $[-h, h]$.

4. To estimate the effect of the viscosity of the liquids adjoining the layer on the reflection and transmission of sound through an inhomogeneous anisotropic elastic layer we calculated numerically the absorption coefficients α, characterizing the loss of energy of the incident sound wave due to the viscosity of the liquids. When the liquids on both sides of the layer are the same we have

$$\alpha = 1 - |A_{11}|^2 - |A_{21}|^2$$

Calculations were carried out for isotropic and two types of transversely isotropic plates, situated in water ($\rho_1 = \rho_2 = 10^3$ kg/m^3, $c_1 = c_2 = 1485$ m/s and $\nu_1 = \nu_2 = 0.13 \times 10^{-5}$ m^2/s). The investigations were made both for the case of homogeneous materials with a density $\rho_0 = 2.7 \times 10^3$ kg/m^3, and for a layer with the following form of inhomogeneity: $\rho(x_3) = \rho_0 f(x_3)$, where $f(x_3)$. In this case the factor a was chosen so that the mean value of the function $f(x_3)$ over the thickness of the layer was unity.

We took the following values of the moduli of elasticity of the anisotropic material for the calculations: $\lambda_{11} = 16.4 \times 10^{10}$ N/m^2, $\lambda_{13} = 3.28 \times 10^{10}$, $\lambda_{33} = 5.74 \times 10^{10}$, $\lambda_{44} = 2.54 \times 10^{10}$ (type 1), and $\lambda_{11} = 5.74 \times 10^{10}$ N/m^2, $\lambda_{13} = 0.819 \times 10^{10}$, $\lambda_{33} = 16.4 \times 10^{10}$ and $\lambda_{44} = 2.95 \times 10^{10}$ (type 2). The value of the modulus of elasticity λ_{12} is not fixed, since, in the case of a transversely isotropic layer, it is not present either in Eqs (2.5) or in boundary conditions (2.13). The chosen materials differ in the fact that the maximum of the phase velocity of quasi-longitudinal waves in a material of type 1 is attained in a direction parallel to the boundaries of the layer, i.e. in the isotropy plane, while in a material of type 2 it is attained in the direction of the axis of elastic symmetry. Here, both the maxima and the minima of the velocities of quasi-longitudinal waves in both cases coincide, while their ratio is 1.7. For the chosen isotropic material $\lambda_{11} = 10.5 \times 10^{10}$ N/m^2, $\lambda_{13} = 5.3 \times 10^{10}$, $\lambda_{33} = 10.5 \times 10^{10}$ and $\lambda_{44} = 2.6 \times 10^{10}$, and it has the same density as the anisotropic materials, and occupies an intermediate position in the velocity of longitudinal waves with respect to the velocity of quasi-longitudinal waves in the anisotropic materials considered.

When making numerical investigations, the solutions of the Cauchy problems (2.5) and (3.1) were found by a fourth-order Runge–Kutta method with an automatic choice of the integration step.

In Fig. 1 we show the coefficient α as a function of the angle of incidence θ_1 of a plane wave for homogeneous and inhomogeneous plates with different types of anisotropy at a fixed frequency of the incident wave ($k_1h = 2.25$). The curves corresponding to the isotropic plates and plates of anisotropic materials of types 1 and 2, are denoted by the numbers 1, 2 and 3 respectively. The results obtained for homogeneous materials are represented by the continuous curves and the results obtained for inhomogeneous materials are represented by the dashed curves.

Note that, for angles of incidence up to $\theta_1 = 75°$, the value of α is small and almost the same for all the materials considered. In the region of $\theta_1 = 90°$ the value of the loss reaches a certain maximum value α_{max} at a certain critical angle θ_*, and then falls to zero when $\theta_1 = 90°$. This qualitative form of the relationship between the absorption coefficient and the angle of incidence is similar in form to that obtained when investigating the reflection of a plane wave from an absolutely rigid surface (the Kontstantinov effect) [7]. This increase in the coefficient α occurs unequally for layers with different types of anisotropy. It occurs to a lesser extent in the type 1 material (for a homogeneous layer the value of α increases to only 0.015). This increase is somewhat greater ($\alpha_{max} = 0.018$) for an isotropic material. The greatest increase is energy loss is observed for the type 2 material ($\alpha_{max} = 0.026$). For different types of materials the values of the critical angles of incidence also turn out to be different. For a homogeneous layer of type 1, $\theta_* = 85.2°$, for an isotropic layer $\theta_* = 86.4°$, and for a layer of type 2, $\theta_* = 87.8°$.

Hence, the effect of anisotropy of the layer manifests itself as follows: type 1 anisotropy somewhat reduces the value of α_{max} and reduces θ_* compared with an isotropic material; type 2 anisotropy increases both the maximum value of the loss and the corresponding critical angle of incidence.

Note that a comparison of the corresponding graphs for homogeneous and inhomogeneous materials shows that they hardly differ when $\theta_1 < 70°$. In the region of the critical angle, for inhomogeneous plates there is a small reduction in the loss compared with the corresponding homogeneous plates. Here the value of α falls by 12–15%. The smaller of these values relates to a type 2 material, and the larger value is for a type 1 material.

The effect of a change in the frequency of the incident wave on the value of the absorption of sound energy and the value of the critical angle can be seen from the graphs in Fig. 2, where we show curves of α against θ_1 for an isotropic inhomogeneous layer for different values of the wave dimensions of the plate: $k_1h = 2.5$, 5, 10 and 20. The graphs show that an increase in the frequency leads to an increase in absorption not only in the region of the critical angles but also at smaller angles of incidence. An increase in frequency shifts the position of the critical angle to lower values.

An analysis of the results of numerical calculations shows that the absorption of acoustic energy due to the viscosity of the surrounding liquids when sound is reflected from and transmitted through an elastic layer in the region of the critical angles depends very much on the type of anisotropy and inhomogeneity of the layer material.

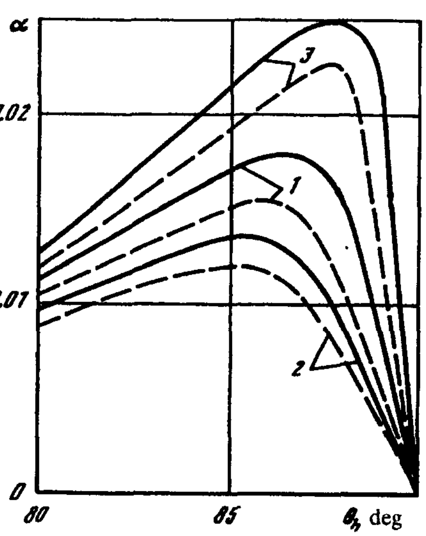

Fig. 1.

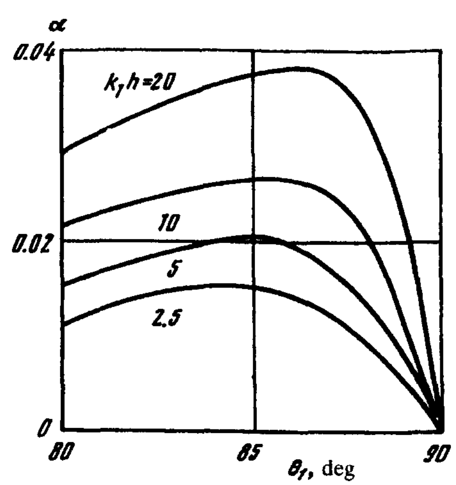

Fig. 2.

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